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MODAL ANALYSIS OF A NON-ROTATING CANTILEVER BEAM USING FINITE ELEMENT METHOD

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Abstract

In this paper finite element method is used to carry out structural dynamic analysis of non-rotating modes of a Cantilever beam. Strain and kinetic energy expressions are derived step by step. Finite element model of the non-rotating cantilever beam is formulated and then analyzed to produce modal response model of the beam. The 2-D beam or *beam* element is developed with the aid of the first theorem of Castigliano. The use of Castigliano's first theorem is for the distinct purpose of introducing the concept of minimum potential energy without resort to the higher mathematic principles of the calculus of variations. The displacement function is discretized to obtain the stiffness matrix and mass matrix for a beam element. The mass matrix, stiffness matrix and force vector are combine to obtain the finite element equations of motion for a beam element. The beam is divided into ten beam element and the summation of the mass matrix and stiffness matrix of each beam element gives the assembled mass matrix and stiffness matrix. Modal analysis of the beam element is carried out to obtain the natural frequencies and mode shapes of the beam element. The results of the finite element model of the non-rotating cantilever beam are validated with the results of NISA 11 packet program.

Keywords - Eigenvalues, Eigenvector, Finite element method, Mode shapes, natural frequency

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1. Introduction

Vibration analysis is one of the vital tasks in designing of structural and mechanical system. The effect of vibration absorber on the rotating machinaries, vehicle suspension system and the dynamic behaviour of machine tool structures due to excitation are the important information that design engineer wants to obtain. This information helps to design system to control the excessive amplitude of the vibration. Utaja and Khairina (2007) studied Dynamic transient analysis of the structure based on the finite element with the normal mode method. Sule (2011) developed a simplified dynamic model to predict the natural frequencies of a multi-Degree of freedom structural system using the concept of an elastic shear wave of a solid prismatic bar. Coskun et al (2011) studied transverse vibration analysis of uniform and non uniform Euler-Bernoulli Beams. Shankar and Harmesh (2012) analyzed structural dynamic analysis of a cantilever beam structure. Chopade and Barjibhe (2013) focuses on the theoretical analysis of transverse vibration of fixed free beam that is a beam that is fixed at one end and free at the other end and investigates the modal shape frequency. Liu and Gurram (2009) used He's Variational Iteration Method to analyze the free vibration of an Euler-Bernoulli beam under various supporting conditions. Hsu et al. (2009) also used Modified Adomian Decomposition Method, a new analytical approximation method, to solve eigenvalue problem for free vibration of uniform Timoshenko beams. Wang (1996) studied the dynamic analysis of generally supported beam. Yieh (1999) determined the natural frequencies and natural modes of the Euler_Bernoulli beam using the singular value decomposition method. Kim (2001) studied the vibration of uniform beams with generally restrained boundary conditions.

In this study, structural dynamics analysis of a cantilever beam will be analyzed. The comparison of the result of the finite element model of the structural dynamic analysis of a cantilever beam constructed by manual assemble of stiffness matrices and mass matrices of ten elements to obtain their natural frequencies and mode shapes and NISA 11 packet program.

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2. Theoretical Consideration

A beam that is fixed at one end and free at the other end is called a cantilever. A cantilever beam of length L, with coordinate system (x, y, z) having the origin 0 is shown in the figure 1.

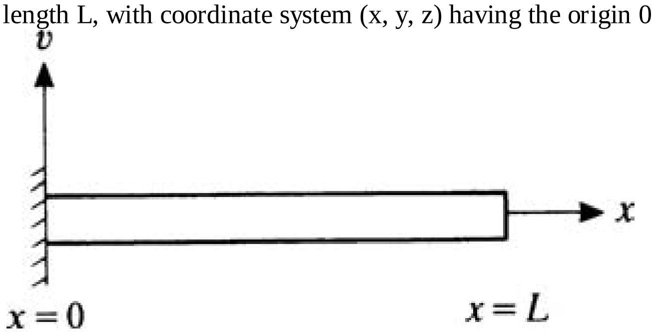

Figure 1. Cantilever beam.

2.1 Derivation of Strain and Kinetic energy expressions

In deriving the energy functions for a bending beam element, it is assumed that the vibration occurs in one of the principal planes of the beam. The beam, which is of length L and has a constant cross-sectional area *A*, is shown in Figure 1. The *xy* plane is the principal plane in which the beam is vibrating and the *x*-axis coincides with the centroidal axis. For wavelengths which are greater than ten times the cross-sectional dimensions of the element, the elementary theory of bending can be used. This theory assumes that the stress components σ_y, σ_z, τ_{yz}, and τ_{xz} are zero. It also assumes that plane sections which are normal to the undeformed centroidal axis remain plane before and after bending and are normal to the deformed axis. With this assumption, the axial displacement, *u*, at a distance *y* from the centroidal axis is

$$u\,(x, y) = -y\,\frac{\partial v}{\partial x} \qquad (1)$$

where $v = v(x)$ is the displacement of the centroidal axis in the *y*-direction at position *x*. The strain components ε_x and y_{xy} are therefore

$$\varepsilon_x = \frac{\partial u}{\partial x} = -y\,\frac{\partial^2 v}{\partial x^2}$$

$$y_{xy} = \frac{\partial u}{\partial y} + \frac{dv}{\partial x} = 0 \qquad (2)$$

The strain energy stored in the element is therefore given by

$$U = \frac{1}{2}\int_{\mathrm{v}} \sigma_x \varepsilon_x \, \mathrm{d}v \qquad (3)$$

The normal stress is given by

$$\sigma_x = E\varepsilon_x \qquad (4)$$

and so equation (3) becomes

$$U = \frac{1}{2}\int_{\mathrm{v}} E\varepsilon^2{}_x \, \mathrm{d}V \qquad (5)$$

Substituting the first of equations (2) into equation (5) gives, since $\mathrm{d}V = \mathrm{d}A \cdot \mathrm{d}x$

$$U = \frac{1}{2}\int_0^L EIz\,\left(\frac{d^2v}{dx^2}\right)^2 dx \qquad (6)$$

Where $\qquad I_z = \int_{\mathrm{A}} y^2 \, \mathrm{d}A \qquad (7)$

is the second moment of area of the cross-section about the *z*-axis.

The stress-strain relations $\tau_{xy} = G\gamma_{xy}$ together with equations (2) suggest that τ_{xy} is zero. In fact this component is nonzero, as can be shown by considering equilibrium. The resulting shear force, *Q*, in the *y*-direction is given by the equation below:

$$\frac{\partial Mz}{\partial x} + Q = 0 \qquad (8)$$

The kinetic energy of a small increment, d*x*, is $\frac{1}{2}\, v^2 \rho A dx$.

The kinetic energy of the complete element is therefore

$$T = \frac{1}{2}\int_0^L \rho A \dot{v}^2 dx \qquad (9)$$

2.2 Finite Element Formulation

The 2-D beam or *beam* element is developed with the aid of the first theorem of Castigliano. The use of Castigliano's first theorem is for the distinct purpose of introducing the concept of minimum potential energy without resort to the higher mathematic principles of the calculus of variations. The nodal variables are the transverse displacements v_1 and v_2 at the nodes and the slopes (rotations) θ_1 and θ_2. The displacement function $v(x)$ is to be discretized such that

$$v(x) = f\,(v_1, v_2, \theta_1, \theta_2, x) \qquad (10)$$

subject to the boundary condition

$$v(x = x_1) = v_1 \qquad (11)$$

$$v(x = x_2) = v_2 \qquad (12)$$

$$\frac{dv}{dx}|_{x=x1} = \theta_1 \tag{13}$$
$$\frac{dv}{dx}|_{x=x2} = \theta_2 \tag{14}$$

Considering the four boundary conditions and the one-dimensional nature of the problem in terms of the independent variable, we assume the displacement function in the form

$$v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{15}$$

Application of the boundary conditions 11 – 15 in succession yields

$$v(x = 0) = v_1 = a_0 \tag{16}$$
$$v(x = L) = v_2 = a_0 + a_1 L + a_2 L^2 + a_3 L^3 \tag{17}$$
$$\frac{dv}{dx}|_{x=0} = \theta_1 = a_1 \tag{18}$$
$$\frac{dv}{dx}|_{x=L} = \theta_2 = a_1 + 2a_2 L + 3a_3 L^2 \tag{19}$$

Equations 16 – 19 are solved simultaneously to obtain the coefficients in terms of the nodal variables as

$$a_0 = v_1 \tag{20}$$
$$a_1 = \theta_1 \tag{21}$$
$$a_2 = \frac{3}{L^2}(v_2 - v_1) - \frac{1}{L}(2\theta_1 + \theta_2) \tag{22}$$
$$a_3 = \frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\theta_2 + \theta_1) \tag{23}$$

Substituting Equations 20, 21, 22 and 23 into Equation 15 and collecting the coefficients of the nodal variables results in the expression

$$v(x) = \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right)v_1 + \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right)\theta_1 + \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right)v_2 + \left(\frac{x^3}{L^2} - \frac{x^2}{L}\right)\theta_2 \tag{24}$$

which is of the form

$$v(x) = N_1(x)v_1 + N_2(x)\theta_1 + N_3(x)v_2 + N_4(x)\theta_2 \tag{25}$$

Applying the first theorem of Castigliano to the strain energy function with respect to nodal displacement v_1 gives the transverse force at node 1 as

$$\frac{\partial U}{\partial v_1} = F_1 = EI_z \int_0^L \left(\frac{d^2N_1}{dx^2}v_1 + \frac{d^2N_2}{dx^2}\theta_1 + \frac{d^2N_3}{dx^2}v_2 + \frac{d^2N_4}{dx^2}\theta_2\right)\frac{d^2N_1}{dx^2}dx \tag{26}$$

while application of the theorem with respect to the rotational displacement gives the moment as

$$\frac{\partial U}{\partial \Theta_1} = M_1 = EI_z \int_0^L \left(\frac{d^2N_1}{dx^2}v_1 + \frac{d^2N_2}{dx^2}\Theta_1 + \frac{d^2N_3}{dx^2}v_2 + \frac{d^2N_4}{dx^2}\Theta_2\right)\frac{d^2N_2}{dx^2}dx \tag{27}$$

For node 2, the results are

$$\frac{\partial U}{\partial v_2} = F_2 = EI_z \int_0^L \left(\frac{d^2N_1}{dx^2}v_1 + \frac{d^2N_2}{dx^2}\Theta_1 + \frac{d^2N_3}{dx^2}v_2 + \frac{d^2N_4}{dx^2}\Theta_2\right)\frac{d^2N_3}{dx^2}dx \tag{28}$$
$$\frac{\partial U}{\partial \Theta_2} = M_2 = EI_z \int_0^L \left(\frac{d^2N_1}{dx^2}v_1 + \frac{d^2N_2}{dx^2}\Theta_1 + \frac{d^2N_3}{dx^2}v_2 + \frac{d^2N_4}{dx^2}\Theta_2\right)\frac{d^2N_4}{dx^2}dx \tag{29}$$

Equations 26 – 29 algebraically relate the four nodal displacement values to the four applied nodal forces (here we use *force* in the general sense to include applied moments) and are of the form

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{34} & K_{44} \end{bmatrix} \begin{Bmatrix} v_1 \\ \Theta_1 \\ v_2 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} \tag{30}$$

By comparison of Equations 26 – 29 with the matrix equation 30, it is seen that

$$K_{mn} = K_{nm} = EI_z \int_0^L \frac{d^2Nm}{dx^2}\frac{d^2Nn}{dx^2}dx \qquad m, n = 1,2,3,4 \tag{31}$$

Integrations of equation 31 become

$$K_{mn} = K_{nm} = EI_z \int_0^L \frac{d^2Nm}{dx^2}\frac{d^2Nn}{dx^2}dx = \frac{EIz}{L^3}\int_0^1 \frac{d^2Nm}{d\xi^2}\frac{d^2Nn}{d\xi^2}d\xi \quad m, n = 1,2,3,4 \tag{32}$$

The stiffness coefficients are then evaluated and the complete stiffness matrix for the element is written as

$$[K^{(e)}] = \frac{EIz}{L^3}\begin{bmatrix} 12 & 6L & -12L & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{33}$$

We now develop the mass matrix for a beam element. We apply Newton's second law of motion to the differential element in the *y* direction to obtain

$$\sum F_y = M_{ay} \Rightarrow V + \frac{\partial v}{\partial x}dx - V - q(x, t)\,dx = (\rho A\,dx)\frac{\partial^2 v}{\partial t^2} \tag{34}$$

Equation 34 simplifies to

$$\frac{\partial v}{\partial x} - q(x, t) = \rho A\frac{\partial^2 v}{\partial t^2} \tag{35}$$

Substituting the moment-shear relation into Equation 35 gives

$$\frac{\partial^2 M}{\partial x^2} - q(x, t) = \rho A\frac{\partial^2 v}{\partial t^2} \tag{36}$$

Under the assumptions of constant elastic modulus *E* and moment of inertia *Iz*, the governing equation for dynamic beam deflection becomes

$$\rho A \frac{\partial^2 v}{\partial t^2} + EI_z \frac{\partial^4 v}{\partial x^4} = -q(x, t) \tag{37}$$

Application of Galerkin's method to equation 37 for a finite element of length L results in the residual equations

$$\int_0^L Ni(x) \left(\rho A \frac{\partial^2 v}{\partial t^2} + EI_z \frac{\partial^4 v}{\partial x^4} + q\right) = 0 \quad i = 1,2,3,4 \tag{38}$$

For each of the four equations represented by Equation 38, the first integral term becomes

$$\rho A \int_0^L Ni\left(N_1\ddot{v}_1 + N_2\ddot{\Theta}_1 + N_3\ddot{v}_2 + N_4\ddot{\Theta}_2\right)dx = \rho A\int_0^L Ni\,[N]dx \begin{Bmatrix} \ddot{v}_1 \\ \ddot{\Theta}_1 \\ \ddot{v}_2 \\ \ddot{\Theta}_2 \end{Bmatrix}$$

$i = 1,2,3,4$ (39)

and, when all four equations are expressed in matrix form, the inertia terms become

$$\rho A\int_0^L N^T [N]\, dx \begin{Bmatrix} \ddot{v}_1 \\ \ddot{\Theta}_1 \\ \ddot{v}_2 \\ \ddot{\Theta}_2 \end{Bmatrix} = [m^{(e)}] \begin{Bmatrix} \ddot{v}_1 \\ \ddot{\Theta}_1 \\ \ddot{v}_2 \\ \ddot{\Theta}_2 \end{Bmatrix} \tag{40}$$

The consistent mass matrix for a two-dimensional beam element is given by

$$[m^{(e)}] = \rho A\int_0^L N^T [N]\, dx \tag{41}$$

Substitution for the interpolation functions and performing the required integrations gives the mass matrix as

$$[m^{(e)}] = \frac{\rho AL}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{42}$$

Combining the mass matrix with previously obtained results for the stiffness matrix and force vector, the finite element equations of motion for a beam element are:

$$m^{(e)} \begin{Bmatrix} \ddot{v}_1 \\ \ddot{\Theta}_1 \\ \ddot{v}_2 \\ \ddot{\Theta}_2 \end{Bmatrix} + K^{(e)} \begin{Bmatrix} v_1 \\ \Theta_1 \\ v_2 \\ \Theta_2 \end{Bmatrix} = -\int_0^L [N]^T q(x, t)\, dx \begin{Bmatrix} -v_1(t) \\ -M_1(t) \\ v_2(t) \\ M_2(t) \end{Bmatrix} \tag{43}$$

The beam is divided into ten equal length parts

1 2 3 4 5 6 7 8 9 10 11 (node numbers)

x----------x----------x----------x----------x----------x----------x----------x----------x----------x----------x

(1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (element numbers)

Figure 2 Ten (10) elements and eleven (11) nodes.

Stiffness matrix

$K_1 = K_2 = K_3 = K_4 = K_5 = K_6 = K_7 = K_8 = K_9 = K_{10} =$

$$\frac{EAI}{l^3} \begin{bmatrix} 12 & 6l & -12l & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{44}$$

The assembled stiffness matrix K is given as

$K = K_1+K_2+K_{3+}K_4+K_5+K_6+K_{7+}K_{8+}K_{9+}K_{10} =$

$$\frac{EIz}{l^3} \left[\begin{array}{cccccccccccccccccccc}
24 & 0 & -12l & 6l \\
0 & 8l^2 & -6l & 2l^2 \\
-12 & -6l & 24 & 0 & -12l & 6l \\
6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\
 & & -12 & -6l & 24 & 0 & -12l & 6l \\
 & & 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\
 & & & & -12 & -6l & 24 & 0 & -12l & 6l \\
 & & & & 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\
 & & & & & & -12 & -6l & 24 & 0 & -12l & 6l \\
 & & & & & & 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\
 & & & & & & & & -12 & -6l & 24 & 0 & -12l & 6L \\
 & & & & & & & & 6l & 2l^2 & 0 & 8l^2 & -6L & 2L^2 \\
 & & & & & & & & & & -12 & -6l & 24 & 0 & -12L & 6L \\
 & & & & & & & & & & 6l & 2l^2 & 0 & 8L^2 & -6L & 2L^2 \\
 & & & & & & & & & & & & -12 & -6L & 24 & 0 & -12L & 6L \\
 & & & & & & & & & & & & 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\
 & & & & & & & & & & & & & & -12 & -6L & 24 & 0 & -12L & 6L \\
 & & & & & & & & & & & & & & 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\
 & & & & & & & & & & & & & & & & -12 & -6L & 12 & -6L \\
 & & & & & & & & & & & & & & & & 6L & 2L^2 & -6L & 4L^2
\end{array}\right] \tag{45}$$

The geometrical and material property of the cantilever beam are given as

Length(L) = 200cm or 80in, Young Modulus(E) = 2.04E6kg/cm^2 or 3.0E7psi, Density(ρ) = 0.2714kg/cm^3 or 7.28E(-4) lbs.sec2/ in4, Area(A) = 25.8cm^2 or 4in^2, Moment of Inertia (I) = 55.5cm^4 or 1.3333 in^4.

Putting in the values of E, I_z and L, the total stiffness matrix or assembled stiffness matrix is

K=10²

$$\begin{bmatrix}
3396.6 & 0 & -1698.3 & 16983 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 452880 & -16983 & 113220 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1698.3 & -16983 & 3396.6 & 0 & -1698.3 & 16983 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
16983 & 113220 & 0 & 452880 & -16983 & 113220 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1698.3 & -16983 & 3396.6 & 0 & -1698.3 & 16983 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 16983 & 113220 & 0 & 452880 & -16983 & 113220 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1698.3 & -16983 & 3396.6 & 0 & -1698.3 & 16983 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 16983 & 113220 & 0 & 452880 & -16983 & 113220 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1698.3 & -16983 & 3396.6 & 0 & -1698.3 & 16983 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 16983 & 113220 & 0 & 452880 & -16983 & 113220 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1698.3 & -16983 & 3396.6 & 0 & -1698.3 & 16983 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16983 & 113220 & 0 & 452880 & -16983 & 113220 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1698.3 & -16983 & 3396.6 & 0 & -1698.3 & 16983 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16983 & 113220 & 0 & 452880 & -16983 & 113220 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1698.3 & -16983 & 3396.6 & 0 & -1698.3 & 16983 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16983 & 113220 & 0 & 452880 & -16983 & 113220 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1698.3 & -16983 & 3396.6 & 0 & -1698.3 & 16983 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16983 & 113220 & 0 & 452880 & -16983 & 113220 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1698.3 & -16983 & 1698.3 & -16983 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16983 & 113220 & -16983 & 226440
\end{bmatrix} \quad (46)$$

Mass Matrix

$$M_1 = M_2 = M_3 = M_4 = M_5 = M_6 = M_7 = M_8 = M_9 = M_{10} = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

(47)

The assembled Mass matrix M is given as

$$M = M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7 + M_8 + M_9 + M_{10} = \frac{\rho AL}{420}$$

$$\begin{bmatrix}
312 & 0 & 54 & -13L & & & & & & & & & & & & & & & & \\
0 & 8L^2 & 13L & -3L^2 & & & & & & & & & & & & & & & & \\
54 & 13L & 312 & 0 & 54 & -13L & & & & & & & & & & & & & & \\
-13L & -3L^2 & 0 & 8L^2 & 13L & -3L^2 & & & & & & & & & & & & & & \\
 & & 54 & 13L & 312 & 0 & 54 & -13L & & & & & & & & & & & & \\
 & & -13L & -3L^2 & 0 & 8L^2 & 13L & -3L^2 & & & & & & & & & & & & \\
 & & & & 54 & 13L & 312 & 0 & 54 & -13L & & & & & & & & & & \\
 & & & & -13L & -3L^2 & 0 & 8L^2 & 13L & -3L^2 & & & & & & & & & & \\
 & & & & & & 54 & 13L & 312 & 0 & 54 & -13L & & & & & & & & \\
 & & & & & & -13L & -3L^2 & 0 & 8L^2 & 13L & -3L^2 & & & & & & & & \\
 & & & & & & & & 54 & 13L & 312 & 0 & 54 & -13L & & & & & & \\
 & & & & & & & & -13L & -3L^2 & 0 & 8L^2 & 13L & -3L^2 & & & & & & \\
 & & & & & & & & & & 54 & 13L & 312 & 0 & 54 & -13L & & & & \\
 & & & & & & & & & & -13L & -3L^2 & 0 & 8L^2 & 13L & -3L^2 & & & & \\
 & & & & & & & & & & & & 54 & 13L & 312 & 0 & 54 & -13L & & \\
 & & & & & & & & & & & & -13L & -3L^2 & 0 & 8L^2 & 13L & -3L^2 & & \\
 & & & & & & & & & & & & & & 54 & 13L & 312 & 0 & 54 & -13L \\
 & & & & & & & & & & & & & & -13L & -3L^2 & 0 & 8L^2 & 13L & -3L^2 \\
 & & & & & & & & & & & & & & & & 54 & 13L & 156 & -22L \\
 & & & & & & & & & & & & & & & & -13L & -3L^2 & -22L & 4L^2
\end{bmatrix} \quad (48)$$

Putting in the values of E, I_z and L, then the total Mass matrix or assembled Mass matrix is

$$M=\begin{bmatrix}
84.6686713329 & 0 & 14.6541931153 & -70.5572261108 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 868.396629056 & 70.5572261108 & -325.648735896 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14.6541931153 & 70.5572261108 & 84.6686713329 & 0 & 14.6541931153 & -70.5572261108 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-70.5572261108 & -325.648735896 & 0 & 868.396629056 & 70.5572261108 & -325.648735896 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 14.6541931153 & 70.5572261108 & 84.6686713329 & 0 & 14.6541931153 & -70.5572261108 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -70.5572261108 & -325.648735896 & 0 & 868.396629056 & 70.5572261108 & -325.648735896 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 14.6541931153 & 70.5572261108 & 84.6686713329 & 0 & 14.6541931153 & -70.5572261108 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -70.5572261108 & -325.648735896 & 0 & 868.396629056 & 70.5572261108 & -325.648735896 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 14.6541931153 & 70.5572261108 & 84.6686713329 & 0 & 14.6541931153 & -70.5572261108 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -70.5572261108 & -325.648735896 & 0 & 868.396629056 & 70.5572261108 & -325.648735896 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14.6541931153 & 70.5572261108 & 84.6686713329 & 0 & 14.6541931153 & -70.5572261108 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -70.5572261108 & -325.648735896 & 0 & 868.396629056 & 70.5572261108 & -325.648735896 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14.6541931153 & 70.5572261108 & 84.6686713329 & 0 & 14.6541931153 & -70.5572261108 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -70.5572261108 & -325.648735896 & 0 & 868.396629056 & 70.5572261108 & -325.648735896 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14.6541931153 & 70.5572261108 & 84.6686713329 & 0 & 14.6541931153 & -70.5572261108 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -70.5572261108 & -325.648735896 & 0 & 868.396629056 & 70.5572261108 & -325.648735896 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14.6541931153 & 70.5572261108 & 84.6686713329 & 0 & 14.6541931153 & -70.5572261108 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -70.5572261108 & -325.648735896 & 0 & 868.396629056 & 70.5572261108 & -325.648735896 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14.6541931153 & 70.5572261108 & 42.3343356664 & -119.404536495 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -70.5572261108 & -325.648735896 & -119.404536495 & 434.198314528
\end{bmatrix} \quad (49)$$

2.3 Formulations for Modal Frequencies of the System

The equation of motion for a multiple degree of freedom undamped structural system is

$$[m]\{\ddot{y}\} + [K]\{y\} = \{F(t)\} \quad (50)$$

In the absence of the external force vector, the equation of motion becomes

$$[m]\{\ddot{y}\} + [K]\{y\} = 0 \quad (51)$$

Under free vibration, the natural frequencies and the mode shapes of a multiple degree of freedom system are the solutions of the eigenvalue problem.

$$[K-\omega^2 m]\ \{\Phi\} = 0 \quad (52)$$

3. Results and Discussion

The eigenvalues and eigenvectors are obtained from equation (77) by putting in the values of the assembled mass matrix and stiffness matrix in the said equation and also evaluate it with Matlab. In order to establish the accuracy of the present study, the results obtained are compared with the results of NISA II.

The first two natural frequencies are extracted and are compared with NISA II and showed in Table 1.

Mode	Present study	NISA II/90
1	10.066 Hz	10.246 Hz
2	64.711 Hz	64.218 Hz

Table 1. natural frequencies of NISA 11 and the present study

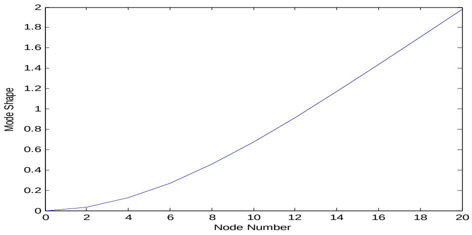

Figure 2 The deformation of 1^{st} mode

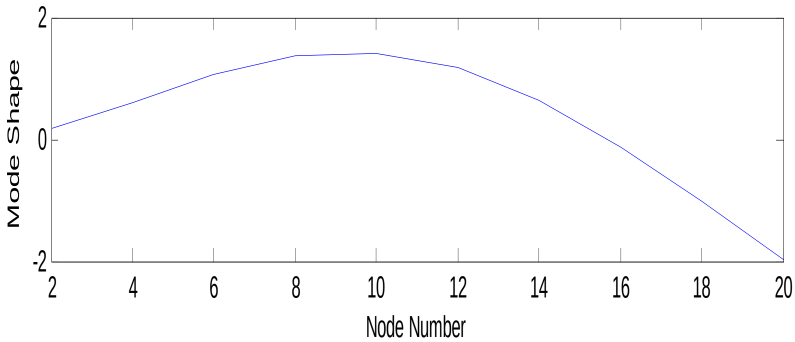

Figure 3. The 2^{nd} mode deformation

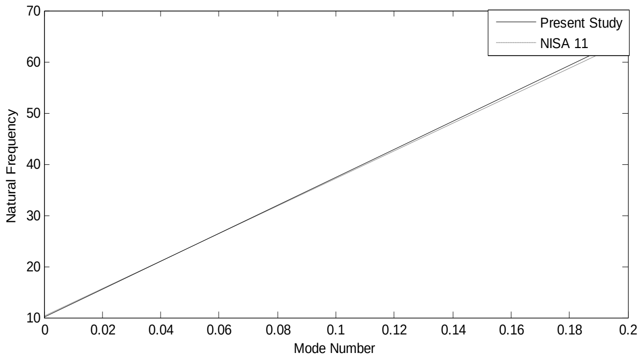

Figure 4. Graph of comparison of the natural frequencies obtained in the present study and NISA 11.

It can be seen from the comparison graph and table1 that the eigenvalues of the present study is almost similar to that of NISA II showing the effectiveness of the present study in the prediction of dynamic response of a vibrating structural system. The difference in responses of the present study and that of NISA II may be due to difference in the beam element divisions that is NISA II used 30 beam elements and 31 nodes while the present study used 10 beam elements and 11 nodes.

4.Conclusion

Modal analysis of a non-rotating cantilever beam has been carried out. In the study ten-beam-finite element model is computed and validated with results of NISA 11 packet program. The results obtained are almost similar with those of NISA 11. The difference in responses of the present model and those of NISA 11 may be due to difference in the number beam element divisions between computation and NISA II. NISA II used 30 beam elements and 31 nodes while the present study used 10 beam elements and 11 nodes.

5. References

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